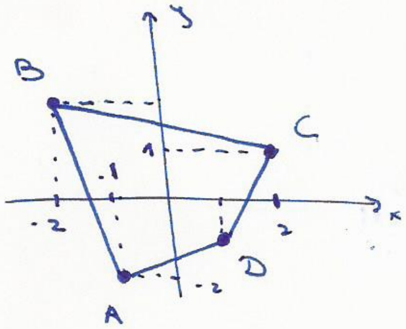
Correction examen MMP L2 du 13/05/2015

Ex. 1



Droite AB :

$$y = -6 - 4x$$

Droite BC :

$$y = \frac{3}{2} - \frac{x}{4}$$

Droite CD :

$$y = 2x - 3$$

Droite AD :

$$y = \frac{x}{2} - \frac{3}{2}$$

D'où on obtient :

$$\iint_Q f(x,y)\,dx\,dy = \int_{-2}^{-1}\left(\int_{-6-4x}^{3/2-\frac{x}{4}} f(x,y)\,dy\right)dx + \int_{-1}^{1}\left(\int_{\frac{x}{2}-\frac{3}{2}}^{3/2-\frac{x}{4}} f(x,y)\,dy\right)dx + \int_{1}^{2}\left(\int_{2x-3}^{3/2-x/4} f(x,y)\,dy\right)dx$$

et par conséquent

$$\text{aire}(Q) = \int_{-2}^{-1}\left(\frac{3}{2}-\frac{x}{4}+6+4x\right)dx + \int_{-1}^{1}\left(\frac{3}{2}-\frac{x}{4}-\frac{x}{2}+\frac{3}{2}\right)dx + \int_{1}^{2}\left(\frac{3}{2}-\frac{x}{4}-2x+3\right)dx = 9.$$

Ex. 2 $x + 2y + 3z = 2015 \Rightarrow$ vecteur normal $\vec{n} = (1,2,3)$

On peut prendre $\vec{E} = \vec{p}$, où $\vec{p}$ note n'importe quel vecteur orthogonal à $\vec{n}$; par exemple, $\vec{p} = (2,-1,0)$

$$\vec{E} = 2\vec{e}_x - \vec{e}_y.$$

Ex. 3

1). $\vec{E} = \cos\frac{\pi x}{4}\sin\frac{\pi y}{4}\,\vec{e}_x + \sin\frac{\pi x}{4}\cos\frac{\pi y}{4}\,\vec{e}_y$

Notons que

$$E_x = \cos\frac{\pi x}{4}\sin\frac{\pi y}{4} = \frac{4}{\pi}\frac{\partial}{\partial x}\sin\frac{\pi x}{4}\sin\frac{\pi y}{4}$$

$$E_y = \sin\frac{\pi x}{4}\cos\frac{\pi y}{4} = \frac{4}{\pi}\frac{\partial}{\partial y}\sin\frac{\pi x}{4}\sin\frac{\pi y}{4}$$

ce qui signifie que $\vec{E}$ est conservatif :

$$\vec{E} = \vec{\nabla} f, \qquad f = \frac{4}{\pi}\sin\frac{\pi x}{4}\sin\frac{\pi y}{4}.$$

2). $$\int_\gamma \vec{E}\cdot d\vec{r} = \int_\gamma \vec{\nabla} f\cdot d\vec{r} = f(B) - f(A) =$$

$$= \frac{4}{\pi}\sin\frac{\pi x}{4}\sin\frac{\pi y}{4}\Bigg|_{(x=1,\,y=1)}^{(x=1,\,y=-1)} =$$

$$= \frac{4}{\pi}\left(-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\cdot\frac{\sqrt{2}}{2}\right) = -\frac{4}{\pi}.$$

Ex. 4

$$\operatorname{div}\vec{E} = 0 \Rightarrow \iint_{\text{demi-sphère}} \vec{E}\cdot d\vec{S} = \iint_{\substack{\text{disque}\\ x^2+y^2\leq 1,\ z=0}} \vec{E}\cdot d\vec{S} =$$

$$= \iint_{\text{disque}} (3\vec{e}_x + 4\vec{e}_y + 5\vec{e}_z)\cdot\vec{e}_z\,dx\,dy =$$

$$= 5\cdot\text{aire(disque)} = 5\pi.$$

Ex. 5

1). $$\int_0^\pi \frac{dx}{(7-\cos x)^2} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{dx}{(7-\cos x)} = \frac{1}{2}\oint_{|z|=1}\frac{dz}{iz\left(7-\frac{z+\bar{z}}{2}\right.}$$

$z_\pm = 7 \pm 4\sqrt{3}$

$$= \frac{2}{i}\oint_{|z|=1}\frac{z\,dz}{(z^2-14z+1)^2} = \frac{2}{i}\cdot 2\pi i\ \operatorname{res}_{z=z_-}\frac{z}{(z-z_+)^2(z-z_-)}$$

$$= 4\pi\left[\frac{d}{dz}\frac{z}{(z-z_+)^2}\right]_{z=z_-} = 4\pi\cdot\frac{7}{768\sqrt{3}} = \frac{7\pi}{192\sqrt{3}}$$

2). $$\int_{-\infty}^{\infty} \frac{e^{-i\pi x}}{x^2+1}\,dx = \left|\begin{matrix}\text{lemme de}\\ \text{Jordan}\end{matrix}\right| = -2\pi i \operatorname*{res}_{x=-i} \frac{e^{-i\pi x}}{x^2+1} =$$

$$= -2\pi i \,\frac{e^{-i\pi(-i)}}{(-i-i)} = \frac{\pi}{e^{\pi}}\,.$$

3). $$\int_{0}^{\infty} \frac{1-\cos x}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\,dx = \frac{1}{2}\int \frac{1-\frac{e^{ix}+e^{-ix}}{2}}{x^2}\,dx$$

$$= \cancel{\frac{1}{2}\int \frac{dx}{x^2}}^{\,0} - \frac{1}{4}\int \frac{e^{ix}}{x^2}\,dx - \cancel{\frac{1}{4}\int \frac{e^{-ix}}{x^2}\,dx}^{\,0}$$

$$= -\frac{1}{4}\cdot 2\pi i \operatorname*{res}_{x=0} \frac{e^{ix}}{x^2} = -\frac{\pi i}{2}\cdot\left[\frac{d}{dx}\,e^{ix}\right]_{x=0} = \frac{\pi}{2}\,.$$